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DIGITAL MODEL CONVERSION OF CONTINUOUS TIME UNCERTAIN SYSTEMS

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Abstract

This paper presents a mixed method consisting of the bilinear approximation and the inverse bilinear approximation method for model conversions of a continuous time structured uncertain linear system to an equivalent discrete time structured uncertain linear model. When the radio of the uncertainties($\Delta A, \Delta B$) to the nominal system parameters (A_0, B_0) and/or the sampling period ($T \le \frac{2}{\|A_0\| + \|\Delta A\|}$) are sufficiently small, the linearized uncertain models can be obtained by neglecting the nonlinear uncertain term $O(T^2 \Delta A \Delta B)$ in the state space models.

Introduction

Most of the real physical dynamical plants are often formulated by a continuous time uncertain framework. The uncertainty about the plant arises from unmodelled dynamics, parameter variation, etc. For digital simulation and digital design of the continuous time uncertain system, The model conversion of a continuous time nominal model has been reported in the literature [1] and [2]. However, the method for model conversion of uncertain state space models has not yet been fully developed. In this paper, we investigate the extension of the bilinear approximation method [2] to uncertain linear systems. In our development, we use a mixed method consisting of the bilinear approximation and inverse bilinear approximation method to carry out the method conversion.

Bilinear and Inverse-Bilinear Approximation Uncertain models

Consider a continuous time uncertain linear system:

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t), \qquad x_c(0) = x_{c0} \tag{1a}$$
$$y_c(t) = C_0 x_c(t) \tag{1b}$$

Where $A = A_0 + \Delta A$, $B = B_0 + \Delta B$

$x_c(t) \in R^{n\times 1}$ is the state, $u_c(t) \in R^{m\times 1}$ is the input, $y_c(t) \in R^{p\times 1}$ is the output, (A_0, B_0, C_0) are nominal system matrices, and ($\Delta A, \Delta B$) is the pair of uncertain matrices which are the perturbations of the nominal system matrices. The associated discrete time uncertain linear model for the system in eqns.1 is

$$x_d(kT+T) = \hat{G}x_d(kT) + \hat{H}u_d(kT), \qquad x_d(0) = x_{c0} \tag{2a}$$
$$y_d(kT) = C_0 x_d(kT) \tag{2b}$$

where
$$\hat{G} = e^{(A_0+\Delta A)T}, \quad \hat{H} = \int_0^T e^{(A_0+\Delta A)t}(B_0 + \Delta B)dt = (\hat{G} - I_n)(A_0 + \Delta A)^{-1}(B_0 + \Delta B),$$
$$u_d(t) = u_d(kT), \qquad kT \le t < (k+1)T$$

T is the sampling period and I_n is an $n \times n$ identity matrix. It is noted that $e^{A_0 T}$ and $e^{\Delta AT}$ are in general not commutative and $\hat{G}$ and $\hat{H}$ consist of nonlinear uncertainty terms in ΔA and ($\Delta A, \Delta B$), respectively. Also note that when $X \in R^{n\times n}$ is a singular matrix and $G = e^{AT}$, then the matrix valued function $(e^{XT} - I_n)X^{-1} = (G - I_n)X^{-1}$ is represented as $\sum_{i=1}^{\infty} T(XT)^{i-1} / i!$.

We need to find the counterpart of the formulation in eqns.1 from eqns.2 . In other words

$$\hat{G} = G_0 + \Delta G \tag{3a}$$

$$\hat{H} = H_0 + \Delta H \tag{3b}$$

Where (G_0, H_0) is the pair of nominal system matrices and $(\Delta G, \Delta H)$ the uncertain system matrices. The matrix-valued function of e^{XT} with $X \in R^{n\times n}$ and a sampling period T can be represented by the bilinear formula [3]

$$G = e^{XT} \cong (I_n - \frac{1}{2}XT)^{-1}(I_n + \frac{1}{2}XT) = -I_n + 2(I_n - \frac{1}{2}XT)^{-1} \stackrel{\Delta}{=} G_1 \qquad T \le \frac{2}{\|X\|} \tag{4a}$$

The inverse bilinear approximations in (4a) for $T \le \frac{2}{\|X\|}$ can be written as

$$(I_n - \frac{1}{2}XT)^{-1}(I_n + \frac{1}{2}XT) = G_1 \cong G = e^{XT} \tag{4b}$$

$$(I_n - \frac{1}{2}XT)^{-1} = (I_n - \frac{1}{2}XT)^{-1}[(I_n + \frac{1}{2}XT) - (I_n - \frac{1}{2}XT)](XT)^{-1} = (G_1 - I_n)(XT)^{-1} \cong (G - I_n)(XT)^{-1} \tag{4c}$$

The inverse bilinear approximations in (4b) and (4c) can be justified by the same reasoning as the bilinear approximation shown in (4a).When X= $A_0 + \Delta A$, the sampling period T for the sufficient condition $\|XT\|/2 < 1$ in (4) can be derived as follows.

$$\because \frac{\|X\|T}{2} = \frac{\|A_0 + \Delta A\|T}{2} \le \frac{(\|A_0\| + \|\Delta A\|)}{2} < 1, \qquad \therefore T < \frac{2}{\|A_0\| + \|\Delta A\|} < \frac{2}{\|A_0\|} \tag{5}$$

Applying the bilinear approximation method in (4) to the $\hat{G}$ in (2) and the G_0 in (3), respectively, result in

$$\hat{G} = e^{(A_0 + \Delta A)T} = e^{AT} \cong (I_n - \frac{1}{2}AT)^{-1}(I_n + \frac{1}{2}AT) \stackrel{\Delta}{=} \hat{G}_1 \tag{6a}$$

$$G_0 = e^{A_0 T} \cong (I_n - \frac{1}{2}A_0 T)^{-1}(I_n + \frac{1}{2}A_0 T) \stackrel{\Delta}{=} G_1 \tag{6b}$$

Thus, the desired discrete-time uncertain system matrix ΔG_1 becomes

$$\Delta G_1 = \hat{G} - G_0 \cong \hat{G}_1 - G_1 = (I_n - \frac{1}{2}AT)^{-1}(I_n + \frac{1}{2}AT) - (I_n - \frac{1}{2}A_0 T)^{-1}(I_n + \frac{1}{2}A_0 T) \tag{7}$$

$$= -I_n + 2(I_n - \frac{1}{2}AT)^{-1} - (-I_n + 2(I_n - \frac{1}{2}A_0 T)^{-1}) = 2(I_n - \frac{1}{2}(A_0 + \Delta A)T))^{-1} - 2(I_n - \frac{1}{2}A_0 T)^{-1}$$

For simplicity, we define U_0 and V_0 as follows:

$$U_0 \stackrel{\Delta}{=} (I_n - \frac{1}{2}A_0 T)^{-1} \tag{8a}$$

$$V_0 = \frac{1}{2}\Delta A T \tag{8b}$$

From (4) we have: $$U_0 \cong (G_0 - I_n)(A_0 T)^{-1} = \frac{1}{2}(G_0 + I_n) \quad (8c)$$

$$\therefore \Delta G_1 \cong 2(U_0^{-1} - V_0)^{-1} - 2U_0 = 2U_0(I_n - V_0 U_0)^{-1} - 2U_0$$

$$\cong 2U_0(I_n + V_0 U_0) - 2U_0 \quad \text{for } \|V_0 U_0\| < 1 \quad (9a)$$

$$= 2U_0 V_0 U_0 = \frac{1}{2}(G_0 - I_n) A_0^{-1} \Delta A (G_0 + I_n) \quad (9b)$$

The discrete time uncertain input matrix ΔH_1 obtained by bilinear and inverse bilinear approximation method can be described as follows. The approximation models of the nominal input H_0 and the uncertain input matrix $\hat{H}$ are represented

$$H_0 = (G_0 - I_n) A_0^{-1} B_0 \cong U_0 B_0 T \overset{\Delta}{=} H_1 \quad (10a)$$

$$\hat{H} = (\hat{G} - I_n) A^{-1} B \cong (I_n - \frac{1}{2}(A_0 + \Delta A)T)(B_0 + \Delta B)T = \hat{H}_1 = (U_0^{-1} - V_0)^{-1}(B_0 + \Delta B)T$$

$$= U_0(I_n - V_0 U_0)^{-1}(B_0 + \Delta B)T \cong U_0(I_n + V_0 U_0)(B_0 + \Delta B)T \quad \text{for } \|V_0 U_0\| < 1$$

$$= U_0 B_0 T + U_0 T(\Delta B + V_0 U_0 B) + O^2(T^2 \Delta A \Delta B) \quad (10b)$$

$$\Delta H_1 \cong \hat{H} - H_0 \cong \hat{H}_1 - H_1 = U_0 T(\Delta B + V_0 U_0 \Delta B) = (G_0 - I_n) A_0^{-1} \Delta B + \frac{1}{2}(G_0 - I_n) A_0^{-1} \Delta A H_0 \quad (11)$$

The linear terms in ΔA ,and ΔB have been retained in the approximation in (11) and the nonlinear term $O(T^2 \Delta A \Delta B)$ in (11) has been neglected. When the value $O(T^2 \Delta A \Delta B)$ is sufficiently small, a better approximate uncertain parameter matrix can be obtained. The sufficient condition $\|V_0 U_0\| < 1$ in (9) and (10) for the existence of $(I_n - V_0 U_0)^{-1} \cong (I_n + V_0 U_0)$ can be verified as follows

$$\because \|V_0 U_0\| \le \|V_0\| \|U_0\| \qquad \therefore \|U_0\| \cong \left\|(I_n - \frac{1}{2} A_0 T)^{-1}\right\| \le \frac{1}{1 - \frac{T}{2}\|A_0\|} \quad \text{for } T < \frac{2}{\|A_0\|} \quad (12a)$$

$$\text{and } \|V_0\| = \frac{T}{2}\|\Delta A\| \quad (12b)$$

$$\therefore \|V_0\| \|U_0\| < \frac{\frac{T}{2}\|\Delta A\|}{1 - \frac{T}{2}\|A_0\|} < 1 \quad (12c)$$

Solving (12c) $$0 < T < \frac{2}{\|A_0\| + \|\Delta A\|} \quad (12d)$$

When the sampling period T satisfies the inequalities (12e), then $(I_n - V_0 U_0)^{-1} \cong (I_n + V_0 U_0)$

Illustrative Example

The unstable dynamics of a helicopter in a vertical plane for an airspeed range of 60~170 knots are given in [4]. The nominal and uncertain system matrices are

$$A_0=\begin{vmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.010 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.707 & 1.3229 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad B_0=\begin{vmatrix} 0.4422 & 0.176 \\ 3.0447 & -7.5922 \\ -5.52 & 4.99 \\ 0 & 0 \end{vmatrix},$$

$$\Delta A=\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \overline{\Delta}_{a1} & 0 & \overline{\Delta}_{a2} \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \Delta B=\begin{vmatrix} 0 & 0 \\ \overline{\Delta}_{b1} & 0 \\ 0 & 0 \\ 0 & 0 \end{vmatrix}$$

$$\left|\overline{\Delta}_{a1}\right|\le 0.2192, \quad \left|\overline{\Delta}_{a2}\right|\le 1.2031 \text{ and } \left|\overline{\Delta}_{b1}\right|\le 2.0673$$

The above perturbation parameters can be normalized as $\Delta_{a1}=\Delta_{a2}=\Delta_{b1}=\Delta_h=\pm 1$. Thus $\Delta A=\Delta_h A_1+\Delta_h A_2$ and $\Delta B=\Delta_h B_1$ and

$$A_1=\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.2192 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad A=\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.2031 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad B=\begin{vmatrix} 0 & 0 \\ 2.0673 & 0 \\ 0 & 0 \\ 0 & 0 \end{vmatrix}$$

Let $T=0.3<\dfrac{2}{\|A_0\|+\|\Delta A\|}$, When $\left|\Delta_h\right|=1\overset{\Delta}{=}\varepsilon$

$$\Delta G=\varepsilon\begin{vmatrix} 0 & -0.00025 & -0.00008 & -0.00138 \\ -0.00002 & -0.00328 & -0.00136 & -0.001836 \\ 0.00088 & 0.05407 & 0.04503 & 0.30864 \\ 0.00008 & 0.00858 & 0.00470 & 0.04833 \end{vmatrix} \quad \Delta H=\varepsilon\begin{vmatrix} 0.00218 & 0.00009 \\ 0.53351 & 0.00151 \\ 0.04061 & -0.04167 \\ 0.00488 & -0.00491 \end{vmatrix}$$

$$\Delta G_1=\varepsilon\begin{vmatrix} 0 & -0.00023 & -0.00018 & -0.00133 \\ -0.00006 & -0.00313 & -0.00237 & -0.01781 \\ 0.00104 & 0.05416 & 0.04098 & 0.30776 \\ 0.00016 & 0.00833 & 0.00631 & 0.04736 \end{vmatrix} \quad \Delta H_1=\varepsilon\begin{vmatrix} 0.00229 & 0.00012 \\ 0.53505 & 0.00162 \\ 0.01095 & -0.2.02801 \\ 0.00055 & -0.00431 \end{vmatrix}$$

$$\Delta H_1=\varepsilon\begin{vmatrix} 0.00229 & 0.00012 \\ 0.53505 & 0.00162 \\ 0.01095 & -0.2.02801 \\ 0.00055 & -0.00431 \end{vmatrix}$$

where $\Delta G=e^{(A_0+\varepsilon A_1+\varepsilon A_2)T}-e^{A_0T}$, $\Delta H=(e^{(A_0+\varepsilon A_1+\varepsilon A_2)T}-I_4)(A_0+\varepsilon A_1+\varepsilon A_2)^{-1}(B_0+\varepsilon B_1)-(e^{A_0T}-I_4)A_0^{-1}B_0$

The associated errors are: $\|\Delta G-\Delta G_1\|/\|\Delta G\|=0.01402$ and $\|\Delta H-\Delta H_1\|/\|\Delta H\|=0.06156$ .The proposed approximate model are quite satisfactory, and the computation load of this method is not heavy.

Conclusions

A mixed method consisting of the bilinear approximation and the inverse bilinear approximation method has been used in this paper to carry out model conversion of a continuous time structured uncertain linear system to an equivalent discrete time structured uncertain model . It is a simple and efficient method. A numerical example is presented to illustrate the proposed method and to demonstrate the effectiveness of the proposed method.

Reference

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